

Reduced Basis Technique for Calculating Sensitivity Coefficients of Nonlinear Structural Response

Ahmed K. Noor* and Jeanne M. Peters†
 NASA Langley Research Center, Hampton, Virginia 23665

An efficient reduced basis technique is presented for calculating the sensitivity of nonlinear structural response to variations in the design variables. The structure is discretized by using two-field mixed finite element models. The vector of structural response and its sensitivity coefficients (derivatives with respect to design variables) are each expressed as a linear combination of a small number of basis (or global approximation) vectors. The Bubnov-Galerkin technique is then used to approximate each of the finite element equations governing the response and the sensitivity coefficients by a small number of algebraic equations in the amplitudes of these vectors. The path derivatives (derivatives of the response vector with respect to path parameters; e.g., load parameters) are used as basis vectors for approximating the response. A combination of the path derivatives and their derivatives with respect to the design variables is used for approximating the sensitivity coefficients. The potential of the proposed technique is discussed and its effectiveness is demonstrated by means of numerical examples of laminated composite plates subjected to mechanical and thermal loads.

Nomenclature

d_i = material and lamination parameters of the plate
 E_L, E_T = elastic moduli of the individual layers in the direction of fibers and normal to it, respectively
 G_{LT}, G_{TT} = shear moduli in the plane of fibers and normal to it, respectively
 $\{G(Z)\}$ = vector of nonlinear terms, see Eqs. (1)
 $\{\hat{G}(\Psi)\}$ = vector of nonlinear terms obtained by replacing $\{Z\}$ in the vector $\{G(Z)\}$ by its expression in terms of $\{\Psi\}$
 $\{\bar{G}(\Psi)\}, \{\bar{G}(\Psi)\}$ = vectors of nonlinear terms of the reduced systems, see Eqs. (5) and (6)
 h = thickness of the plate
 $[K]$ = global linear structure matrix which includes the flexibility and the linear strain-displacement matrices, see Eqs. (1) and (A2)
 $[\bar{K}], [\bar{K}]$ = linear matrices of the reduced systems, see Eqs. (5) and (6)
 L = side length of the plate
 \bar{N}_1 = uniform edge compressive loading, see Fig. 1
 p_0 = intensity of uniform transverse loading
 q_1, q_2 = mechanical load and temperature parameters
 $\{Q^{(1)}\}, \{Q^{(2)}\}$ = normalized applied mechanical load and thermal strain vectors, see Eqs. (1)
 $\{\bar{Q}^{(1)}\}, \{\bar{Q}^{(2)}\}$ = vectors of mechanical load and thermal strain of the reduced systems, see Eqs. (5) and (6)
 $\{\hat{Q}^{(1)}\}, \{\hat{Q}^{(2)}\}$ = vectors of mechanical load and thermal strain of the reduced systems, see Eqs. (5) and (6)

r = number of basis vectors used in evaluating the response
 T_0 = uniform temperature increase
 U = total strain energy of the plate
 u_1, u_2, w = displacement components in the coordinate directions
 x_1, x_2, x_3 = Cartesian coordinate system (x_3 normal to the middle plane of the plate)
 $\{Z\}$ = response vector that includes both nodal displacements and stress parameters
 α_L, α_T = coefficients of thermal expansion of the individual layers in the direction of fibers and normal to it, respectively
 $[\Gamma], [\bar{\Gamma}]$ = matrices of basis vectors, see Eqs. (3) and (4)
 θ = fiber orientation angle for individual layers
 ν_{LT} = major Poisson's ratio of the individual layers
 ϕ_1, ϕ_2 = rotation components of the middle plane of the plate, see Fig. 1
 $\{\Psi\}, \{\bar{\Psi}_i\}$ = vectors of undetermined coefficients of the reduced equations, see Eqs. (5) and (6)
Superscripts
 t = matrix transposition
Range of Indices
 I, J = 1 to total number of degrees of freedom (nodal displacements and stress parameters)
 i, j = 1 to total number of reduced degrees of freedom used in evaluating the sensitivity coefficients
 ι = 1 to the total number of lamination and material parameters

Received Aug. 13, 1991; revision received Dec. 5, 1991; accepted for publication Dec. 6, 1991. Copyright © 1991 by the American Institute of Aeronautics and Astronautics, Inc. No copyright is asserted in the United States under Title 17, U.S. Code. The U.S. Government has a royalty-free license to exercise all rights under the copyright claimed herein for Governmental purposes. All other rights are reserved by the copyright owner.

*Ferman W. Perry Professor of Aerospace Structures and Applied Mechanics, and Director, Center for Computational Structures Technology, University of Virginia. Fellow AIAA.

†Senior Programmer Analyst, Center for Computational Structures Technology, University of Virginia.

Introduction

THE continuing active development of nonlinear analysis procedures and the increasing use of these procedures in automated optimum design of structural systems has stimulated interest in the development of efficient techniques for calculating the sensitivity of nonlinear structural response to variations in the design variables (see, for example, Refs.

1–10, and the survey papers, Refs. 11–14). The sensitivity coefficients (derivatives of the response vector with respect to design variables) are used to: 1) determine a search direction in the direct application of nonlinear mathematical programming algorithms. When approximation concepts are used for the optimum design of structures, sensitivity coefficients are used to construct explicit approximations for the critical, and potentially critical, behavior constraints; 2) generate an approximation for the response of a modified structure (in conjunction with a reanalysis technique); 3) assess the effects of uncertainties in the material and geometric parameters of the computational model on the structural response; and 4) predict the changes in the structural response due to changes in the parameters.

Two general procedures are currently being used for calculating the sensitivity coefficients of the nonlinear structural response: 1) the direct differentiation method; and 2) the adjoint variable method (see, for example, Refs. 3, 8–11). The first procedure is based on the implicit differentiation of the equations that describe the nonlinear structural response with respect to the desired parameters, and the solution of the resulting sensitivity equations. In the adjoint variable method an adjoint physical system is introduced whose solution permits rapid evaluation of the desired sensitivity coefficients. Both procedures can be applied to either the governing discrete equations or to the functional of the variational formulation of the structure (with a consequent change in the order of discretization and implicit differentiation).

The discrete models for complex structures typically have large number of degrees of freedom, and the calculation of the sensitivity coefficients of the nonlinear structural response can become quite expensive. Although efficient reduced basis techniques have been developed for substantially reducing the number of degrees of freedom in the initial discretization and the computational effort involved in nonlinear structural analysis (see, for example, Refs. 15–17), no attempt has been made to adapt these techniques to the calculation of sensitivity coefficients. The present study is an attempt to fill this void. Specifically, the objective of this paper is to present a reduced basis technique and a computational procedure for the efficient calculation of sensitivity of the nonlinear structural response to variations in the design variables. The crux of the technique is the effective choice of basis vectors for approximating the sensitivity coefficients.

To sharpen the focus of the study, discussion is limited to multilayered composite plates with geometric nonlinearities. The design variables consist of lamination and material properties. Both mechanical and thermal loads are considered.

Mathematical Formulation

Governing Finite Element Equations

The analytical formulation is based on a form of the moderate rotation, geometrically nonlinear theory of the structure. A total Lagrangian formulation is used for describing the deformation and the structure is discretized by using two-field mixed finite element models. The governing finite element equations for the response vector and its sensitivity coefficients can be written in the following compact form.^{10,15,17}

$$[K]\{Z\} + \{G(Z)\} - q_1\{\bar{Q}^{(1)}\} - q_2\{\bar{Q}^{(2)}\} = 0 \quad (1)$$

and

$$\begin{aligned} \left[[K] + \left[\frac{\partial G_I}{\partial Z_J} \right] \right] \left\{ \frac{\partial Z}{\partial d_i} \right\} &= - \left[\frac{\partial K}{\partial d_i} \right] \{Z\} - \left\{ \frac{\partial G}{\partial d_i} \right\} \\ &+ q_1 \left\{ \frac{\partial \bar{Q}^{(1)}}{\partial d_i} \right\} + q_2 \left\{ \frac{\partial \bar{Q}^{(2)}}{\partial d_i} \right\} \end{aligned} \quad (2)$$

where $[K]$ is a global linear structure matrix that includes the flexibility, and the linear strain-displacement matrices; $\{Z\}$ is the response vector that includes both nodal displacements and stress parameters; $\{G(Z)\}$ is the vector of nonlinear terms; q_1 and q_2 are mechanical load and temperature parameters; $\{\bar{Q}^{(1)}\}$ and $\{\bar{Q}^{(2)}\}$ are normalized applied mechanical load and thermal strain vectors; and d_i are design variables, selected in the present study to be lamination and material parameters. The form of the arrays $[K]$, $\{G(Z)\}$, $\{\bar{Q}^{(1)}\}$ and $\{\bar{Q}^{(2)}\}$ is described in Appendix A. Note that Eqs. (1) are nonlinear in $\{Z\}$ but Eqs. (2) are linear in $\{\partial Z/\partial d_i\}$.

Basis Reduction and Reduced System of Equations

The response vector $\{Z\}$ and its derivatives with respect to d_i , $\{\partial Z/\partial d_i\}$, are expressed as linear combinations of a few preselected basis vectors. The approximation can be expressed by the following transformation

$$\{Z\} = [\Gamma]\{\Psi\} \quad (3)$$

and

$$\left\{ \frac{\partial Z}{\partial d_i} \right\} = [\Gamma] \left\{ \bar{\Psi}_i \right\} \quad (4)$$

The columns of the matrices $[\Gamma]$ and $[\bar{\Gamma}_i]$ in Eqs. (3) and (4) are the basis vectors, and the elements of the vectors $\{\Psi\}$ and $\{\bar{\Psi}_i\}$ are the amplitudes of the approximation vectors which are, as yet, unknowns. Note that the number of basis vectors in Eqs. (3) and (4) is considerably smaller than the total number of degrees of freedom (components of the vectors $\{Z\}$ and $\{\partial Z/\partial d_i\}$).

A Bubnov-Galerkin technique is now used to replace the governing equations for the structural response and its sensitivity coefficients, Eqs. (1) and (2), by the following reduced equations in the unknowns $\{\Psi\}$ and $\{\bar{\Psi}_i\}$

$$[\bar{K}]\{\Psi\} + \{\bar{G}(\Psi)\} - q_1\{\bar{Q}^{(1)}\} - q_2\{\bar{Q}^{(2)}\} = 0 \quad (5)$$

and

$$\begin{aligned} \left[[\bar{K}] + \left[\frac{\partial \bar{G}_I}{\partial \bar{\Psi}_J} \right] \right] \{\bar{\Psi}_i\} &= - \left[\frac{\partial \bar{K}}{\partial d_i} \right] \{\Psi\} - \left\{ \frac{\partial \bar{G}}{\partial d_i} \right\} \\ &+ q_1 \left\{ \frac{\partial \bar{Q}^{(1)}}{\partial d_i} \right\} + q_2 \left\{ \frac{\partial \bar{Q}^{(2)}}{\partial d_i} \right\} \end{aligned} \quad (6)$$

where

$$[\bar{K}] = [\Gamma][K][\Gamma] \quad (7)$$

$$\{\bar{G}(\Psi)\} = [\Gamma]\{G(\Psi)\} \quad (8)$$

$$\{\bar{Q}^{(1)}\} = [\Gamma]\{Q^{(1)}\} \quad (9)$$

$$\{\bar{Q}^{(2)}\} = [\Gamma]\{Q^{(2)}\} \quad (10)$$

$$[\bar{K}] = [\bar{\Gamma}_i][K][\bar{\Gamma}_i] \quad (11)$$

$$\left[\frac{\partial \bar{G}_I}{\partial \bar{\Psi}_J} \right] = [\bar{\Gamma}_i]' \left[\frac{\partial G_I}{\partial Z_J} \right] [\bar{\Gamma}_i] \quad (12)$$

$$\left[\frac{\partial \bar{K}}{\partial d_i} \right] = [\bar{\Gamma}_i]' \left[\frac{\partial K}{\partial d_i} \right] [\bar{\Gamma}_i] \quad (13)$$

$$\left\{ \frac{\partial \bar{G}}{\partial d_i} \right\} = [\bar{\Gamma}_i]' \left\{ \frac{\partial G}{\partial d_i} \right\} \quad (14)$$

$$\left\{ \frac{\partial \bar{Q}^{(1)}}{\partial d_i} \right\} = [\bar{\Gamma}_i]' \left\{ \frac{\partial Q^{(1)}}{\partial d_i} \right\} \quad (15)$$

$$\left\{ \frac{\partial \bar{Q}^{(2)}}{\partial d_i} \right\} = [\bar{\Gamma}_i] \left\{ \frac{\partial Q^{(2)}}{\partial d_i} \right\} \quad (16)$$

Note that Eqs. (5) are nonlinear in the unknowns $\{\Psi\}$ and Eqs. (6) are linear in $\{\bar{\Psi}_i\}$. The vector $\{G(\Psi)\}$ in Eqs. (8) is obtained by replacing $\{Z\}$ in the vector $\{G(Z)\}$ by its expression in terms of $\{\Psi\}$, Eqs. (3).

Selection and Generation of Basis Vectors

The effectiveness of the proposed technique for calculating the sensitivity coefficients depends, to a great extent, on the proper choice of the basis vectors, the columns of the matrices $[\Gamma]$ and $[\bar{\Gamma}_i]$. An effective choice for the basis vectors used in approximating the response vector $\{Z\}$, Eqs. (3), was found to be the various order path derivatives (derivatives with respect to the control parameters q_1 and q_2);^{16,17} that is, the matrix $[\Gamma]$ used in approximating $\{Z\}$, over a range of values of q_1 and q_2 , includes the response vector corresponding to a particular combination of q_1, q_2 (viz, q_1^0, q_2^0) and its various-order derivatives with respect to q_1 and q_2 , evaluated at the same values of q_1^0, q_2^0 , or

$$[\Gamma] = \left[\begin{array}{c} \{Z\} \left\{ \frac{\partial Z}{\partial q_1} \right\} \left\{ \frac{\partial Z}{\partial q_2} \right\} \left\{ \frac{\partial^2 Z}{\partial q_1^2} \right\} \left\{ \frac{\partial^2 Z}{\partial q_1 \partial q_2} \right\} \left\{ \frac{\partial^2 Z}{\partial q_2^2} \right\} \dots \end{array} \right]_{q_1^0, q_2^0} \quad (17)$$

The basis vectors for approximating the sensitivity coefficients $\{\partial Z/\partial d_i\}$ were selected to be a combination of: 1) the various order path derivatives; that is, the columns of the matrix $[\Gamma]$; and 2) the derivatives of $[\Gamma]$ with respect to d_i ; that is $[\partial \Gamma/\partial d_i]$. Therefore

$$[\bar{\Gamma}_i] = \left[[\Gamma] \left[\frac{\partial \Gamma}{\partial d_i} \right] \right] \quad (18)$$

and the number of basis vectors in $[\bar{\Gamma}_i]$, for each d_i , is twice the number of basis vectors in $[\Gamma]$.

The rationale for the particular choice of $[\bar{\Gamma}_i]$ is based on the following two facts: 1) differentiating Eqs. (3) with respect to d_i leads to:

$$\left\{ \frac{\partial Z}{\partial d_i} \right\} = [\Gamma] \left\{ \frac{\partial \Psi}{\partial d_i} \right\} + \left[\frac{\partial \Gamma}{\partial d_i} \right] \{\Psi\} \quad (19)$$

that is, the expression for $\{\partial Z/\partial d_i\}$ includes both $[\Gamma]$ and $[\partial \Gamma/\partial d_i]$; and 2) the use of free parameters $\{\Psi_i\}$, instead of the fixed amplitudes $\{\partial \Psi/\partial d_i\}$ and $\{\Psi\}$ is expected to improve the accuracy of the approximation for $\{\partial Z/\partial d_i\}$, over a wide range of values of (q_1, q_2) . The free parameters $\{\Psi_i\}$ are obtained by applying the Bubnov-Galerkin technique to Eqs. (2), resulting in Eqs. (6).

The path derivatives (columns of the matrix $[\Gamma]$) are obtained by successive differentiation of the governing finite element equations of the structure, Eqs. (1), with respect to the parameters q_1 and q_2 . The recursion relations for evaluating the path derivatives are given in Ref. 18. The first few recursion relations are summarized in Appendix B. Note that *only one matrix factorization is needed for generating all the path derivatives*.

The derivatives of $[\Gamma]$ with respect to d_i , $[\partial \Gamma/\partial d_i]$, are obtained by differentiating each of the recursion relations for evaluating the path derivatives with respect to d_i . The resulting equations have the same left-hand side as that of the recursion relations, and therefore, *no additional matrix factorizations are needed for generating $[\partial \Gamma/\partial d_i]$* (see Appendix B).

Computational Procedure

The computational procedure for generating the nonlinear response vector $\{Z\}$ and its sensitivity coefficients, $\{\partial Z/\partial d_i\}$, can be conveniently divided into two distinct phases, namely: 1) evaluation of the basis vectors, at a particular combination of q_1, q_2 (viz., q_1^0, q_2^0) and generation of the reduced equations; and, 2) marching with the reduced equations in the solution space and generating the response and sensitivity coefficients at different combinations of q_1, q_2 .

For each combination of q_1 and q_2 , the vector of reduced unknowns, $\{\Psi\}$, is obtained by solving the reduced nonlinear equations, Eqs. (5). Then the vectors $\{\bar{\Psi}_i\}$, associated with the same value of $\{\Psi\}$, are evaluated by solving the reduced linear equations, Eqs. (6). The response vector and its sensitivity coefficients, $\{Z\}$ and $\{\partial Z/\partial d_i\}$, are obtained by using Eqs. (3) and (4). The process is repeated for different combinations of q_1 and q_2 .

The computational procedure proposed in Refs. 16, 17, and 18 for sensing and controlling the error of the response vector $\{Z\}$, predicted by the reduced system of equations, is used for error sensing and control of the sensitivity coefficients $\{\partial Z/\partial d_i\}$.

Numerical Studies

To test and evaluate the effectiveness of the proposed reduced basis technique, the sensitivity coefficients of the nonlinear and postbuckling responses of several structures were generated by this technique. Comparisons were made with the sensitivity coefficients obtained by using the full system of equations of the finite element model. Typical results are presented herein for the simply supported eight-layer quasi-isotropic composite plate shown in Fig. 1. The sensitivity coefficients are calculated for the two cases: 1) nonlinear response due to uniform transverse loading ($q_1 = p_0, q_2 = 0$); and 2) postbifurcation response when the plate is subjected to uniform axial compressive edge loading, \bar{N}_1 , and a uniform temperature increase T_0 ($q_1 = \bar{N}_1, q_2 = T_0$).

Both the nonlinear and postbifurcation responses of the plate exhibit inversion symmetry characterized by the following relations:

$$\left. \begin{array}{l} u_\alpha(x_1, x_2) = -u_\alpha(-x_1, -x_2) \\ w(x_1, x_2) = w(-x_1, -x_2) \\ \phi_\alpha(x_1, x_2) = -\phi_\alpha(-x_1, -x_2) \end{array} \right\} \alpha = 1, 2 \quad (20)$$

The above symmetry relations were used in conjunction with the procedure outlined in Ref. 19 to reduce the size of the analysis model to one-half of the plate. Note that the plate responses do not exhibit reflection symmetry with respect to the x_1 and x_2 planes, and therefore, the analysis model cannot

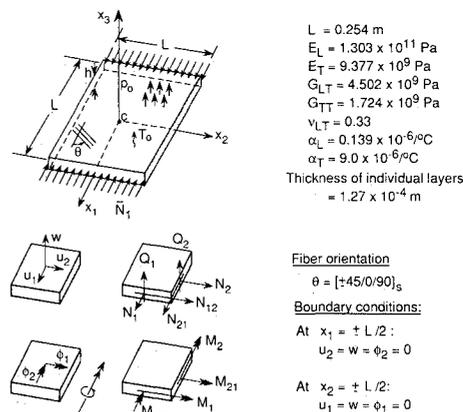


Fig. 1 Composite plate considered in the present study and sign convention for generalized displacements and stress resultants.

be reduced to one quadrant of the plate. An 8×16 grid of mixed finite elements was used for the discretization of half the plate. Biquadratic shape functions were used for approximating each of the generalized displacements, and bilinear shape functions were used for approximating each of the stress resultants. The total number of stress resultant parameters was 4096, and the total number of generalized displacement degrees of freedom was 2525.

The characteristics of the finite element model are given in Ref. 20. The nonlinear and postbuckling responses as well as their sensitivity coefficients were calculated by using the reduced basis technique. Typical results are presented in Figs. 2, 3, and 4 for the nonlinear response and in Figs. 5, 6, and 7 for the postbuckling response. The results are discussed subsequently.

Nonlinear Response Due to Uniform Transverse Loading

Figure 2 shows plots of the loading vs the maximum transverse displacement w_c and the total strain energy U as well as their sensitivity coefficients with respect to the material parameters, $E_L, E_T, G_{LT}, \nu_{LT}, G_{TT}$ and the total plate thickness h . As can be seen from Fig. 2, both w_c and U are very sensitive to variations in h and E_L ; they are slightly sensitive to variations in E_T and G_{LT} and fairly insensitive to variations in ν_{LT} and G_{TT} .

The basis vectors for both the response and the sensitivity coefficients were generated for the unloaded plate $p_0 = 0$, and were thus, obtained by solving a linear system of finite element equations. Then the reduced equations for evaluating the response and the sensitivity coefficients were generated. The basis vectors were not updated throughout the range of loading considered. An indication of the accuracy of w_c and U , and their sensitivity coefficients, obtained by the reduced basis technique is given in Fig. 3.

For the range of loading considered, w_c and U obtained by using ten basis vectors are almost indistinguishable from those obtained by using the full system of finite element equations. The sensitivity coefficients of w_c and U with respect to h, E_L , and G_{LT} obtained by using twenty basis vectors (first ten derivatives of $\{Z\}$ with respect to p_0 , and their first derivatives with respect to $d_r, r = 10$) are as accurate as w_c and U obtained by using ten basis vectors (first ten derivatives of $\{Z\}$ with

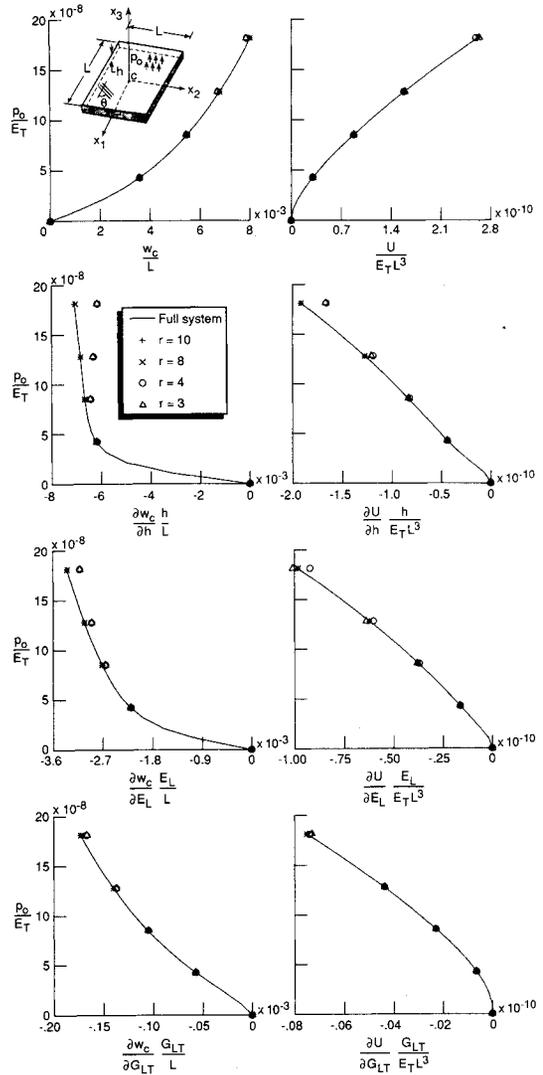


Fig. 3 Accuracy of the nonlinear response and sensitivity coefficients obtained by the reduced basis technique. Composite plate subjected to uniform normal loading (see Fig. 1).

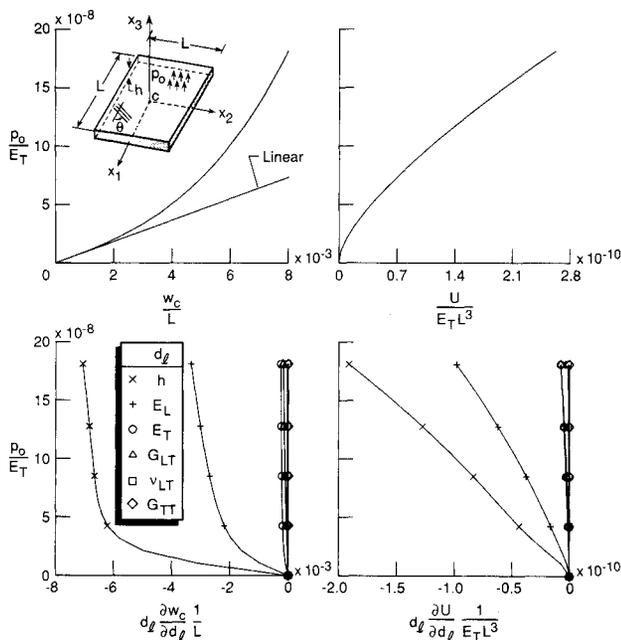


Fig. 2 Nonlinear response and sensitivity coefficients for the composite plate subjected to uniform normal loading (see Fig. 1).

respect to $p_0, r = 10$). Normalized contour plots for w, ϕ_1, ϕ_2 and their sensitivity coefficients with respect to h and E_L , at $p_0/E_T = 18.13 \times 10^{-8}$, are shown in Fig. 4. Note that the contour plots of the generalized displacements bear some resemblance to the contour plots of their sensitivity coefficients. Also note that the contour lines for the sensitivity coefficients are somewhat closer to each other than the corresponding ones for the response functions. This supports the observation, which has been mentioned in Ref. 21, that accurate sensitivity calculations may require a more refined grid than accurate response calculations.

The computational time associated with the foregoing technique is considerably less than that associated with the direct application of Eqs. (2). This is particularly true when the sensitivity coefficients are needed at several different values of p_0 . This is because the decomposed full-structure matrix on the left-hand-side of Eqs. (2) is needed for each value of p_0 at which the sensitivity coefficients are required. Unless the response vector $\{Z\}$ is obtained by solving the full system of nonlinear equations, Eqs. (1), using the Newton-Raphson technique the decomposed left-hand-side matrix of Eqs. (2) is not readily available (which is the case when the response vector is obtained by using quasi-Newton method or the reduced basis technique). By contrast, in the foregoing technique, the most time-consuming operations are those associated with the generation of the basis vectors and reduced

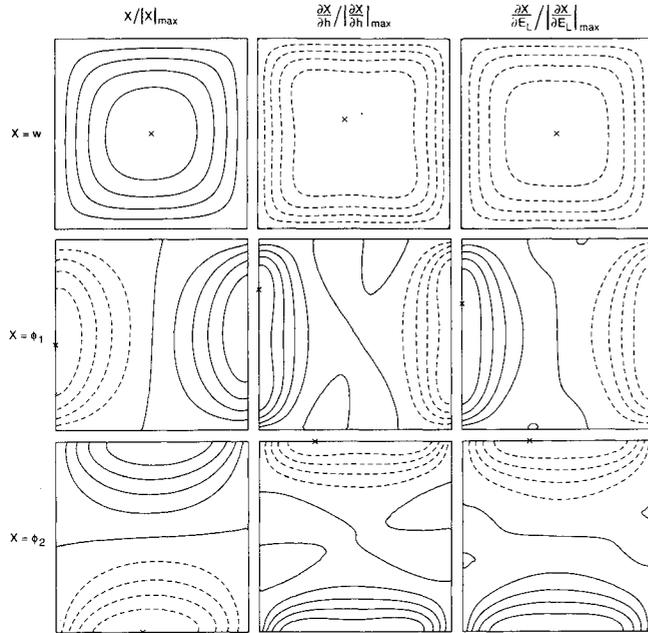


Fig. 4 Normalized contour plots for the generalized displacement components and their sensitivity coefficients. Composite plate subjected to uniform normal loading (see Fig. 1). Spacing of contour lines is 0.2 and dashed lines denote negative contours. An \times denotes location of maximum absolute value of the component. $p_0/E_T = 18.13 \times 10^{-8}$.

equations. The computational time expended in the solution of the reduced equations for the response and the sensitivity coefficients is very small. Because the basis vectors are evaluated at $p_0 = 0$, their generation requires only the decomposition of the linear matrix of the plate $[K]$; evaluation of the right-hand sides of the recursion relations for the basis vectors, and forward reduction-back substitution. These operations are computationally less expensive than the decompositions of the left-hand side matrices of Eqs. (2). For the present problem, the computer time required to generate the sensitivity derivatives using Eqs. (2) at the different values of p_0 , after the response vectors $\{Z\}$ were obtained using the reduced basis technique, was 4.64 times that of the foregoing technique. This ratio increases rapidly as the number of times of evaluating the sensitivity derivatives increases.

Postbuckling Response Due to Combined Edge Compressive Loading and Uniform Temperature Rise

Figure 5 shows the stability boundary (interaction curve), the postbuckling response curves for various values of the applied edge compressive loading; and the sensitivity coefficients of the normal displacement at the center w_c and the total strain energy U . The material parameters considered are E_L , E_T , G_{LT} , α_L , and α_T . The interaction curve is nearly a straight line. The response curves, associated with different values of the edge compressive loading, are nearly parallel.

The center displacement w_c immediately after buckling, is very sensitive to changes in E_L , α_T , and E_T . However, for higher values of T_0 , w_c is fairly insensitive to changes in these material properties. The total strain energy U is very sensitive to variations in E_L , E_T , and α_T and less sensitive to variations in G_{LT} and α_L . The sensitivity coefficients of U increase with increasing temperature.

A nonlinear solution was obtained in the vicinity of the stability boundary for $\bar{N}_1/\bar{N}_{1cr} = 0.50$. Newton-Raphson technique was used in conjunction with the full system of equations to generate the solution. The basis vectors for evaluating both the postbuckling response and its sensitivity coefficients were generated at the same value of T_0 . The left-hand side of the equations used in evaluating the basis vectors are the

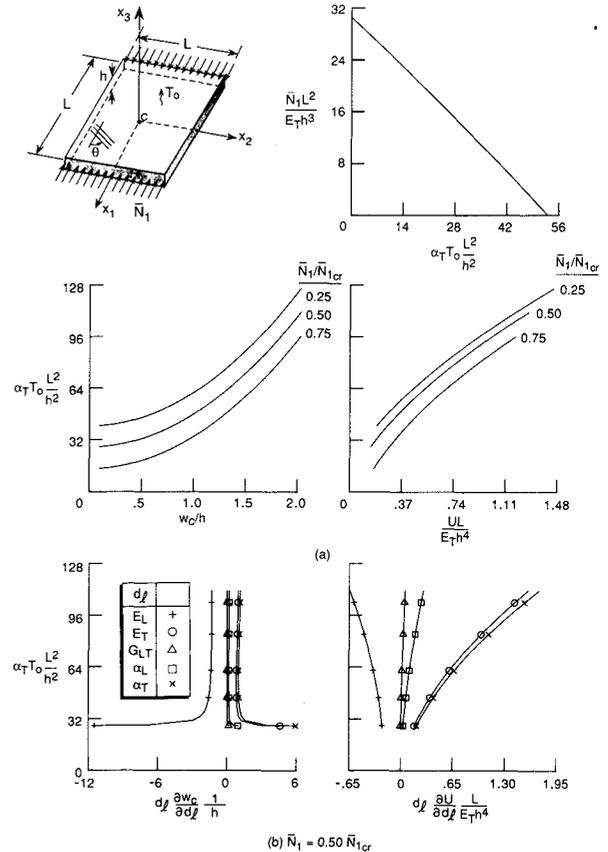


Fig. 5 Stability boundary and postbuckling response of the composite plate subjected to combined temperature increase and axial edge compression (see Fig. 1). $\bar{N}_{1cr} L^2/E_T h^3 = 30.85$.

same as those in the last Newton-Raphson iteration used in evaluating the nonlinear postbuckling response.

An indication of the accuracy of w_c and U and their sensitivity coefficients obtained by using the reduced basis technique is given in Fig. 6. Note that, for the range of temperature considered, w_c and U obtained by using ten basis vectors (the nonlinear solution and its first nine derivatives with respect to T_0) are indistinguishable from those obtained using the full system of finite element equations. The same is true about the sensitivity coefficients of w_c and U obtained by using twenty basis vectors (the ten basis vectors used for approximating the response and their first derivatives with respect to d_i).

Normalized contour plots for the generalized displacements w , ϕ_1 , ϕ_2 and their sensitivity coefficients with respect to E_L and α_T are shown in Fig. 7 for two values of T_0 . At $T_0 = 27.74 h^2/(\alpha_T L^2)$, which corresponds to a plate configuration in the vicinity of buckling, the contour plots for the sensitivity coefficients are very similar to those of the response quantities. For higher temperatures, for example, $T_0 = 111.41 h^2/(\alpha_T L^2)$, the contour plots for the sensitivity derivatives of w are slightly different, from the corresponding w contours.

Potential of the Foregoing Reduced Basis Technique

The foregoing reduced basis technique seems to have high potential for use in large-scale automated analysis and design systems. The numerical studies conducted clearly demonstrate the accuracy and effectiveness of the technique. In particular, the following comments seem to be in order.

1) The particular choice of the basis vectors used herein for approximating the sensitivity coefficients (columns of the matrix $[\bar{\Gamma}_i]$) allows the accurate prediction of these coefficients for a wide range of values of the load parameters.

2) The computational procedure developed for predicting the nonlinear response using the reduced basis technique (see,

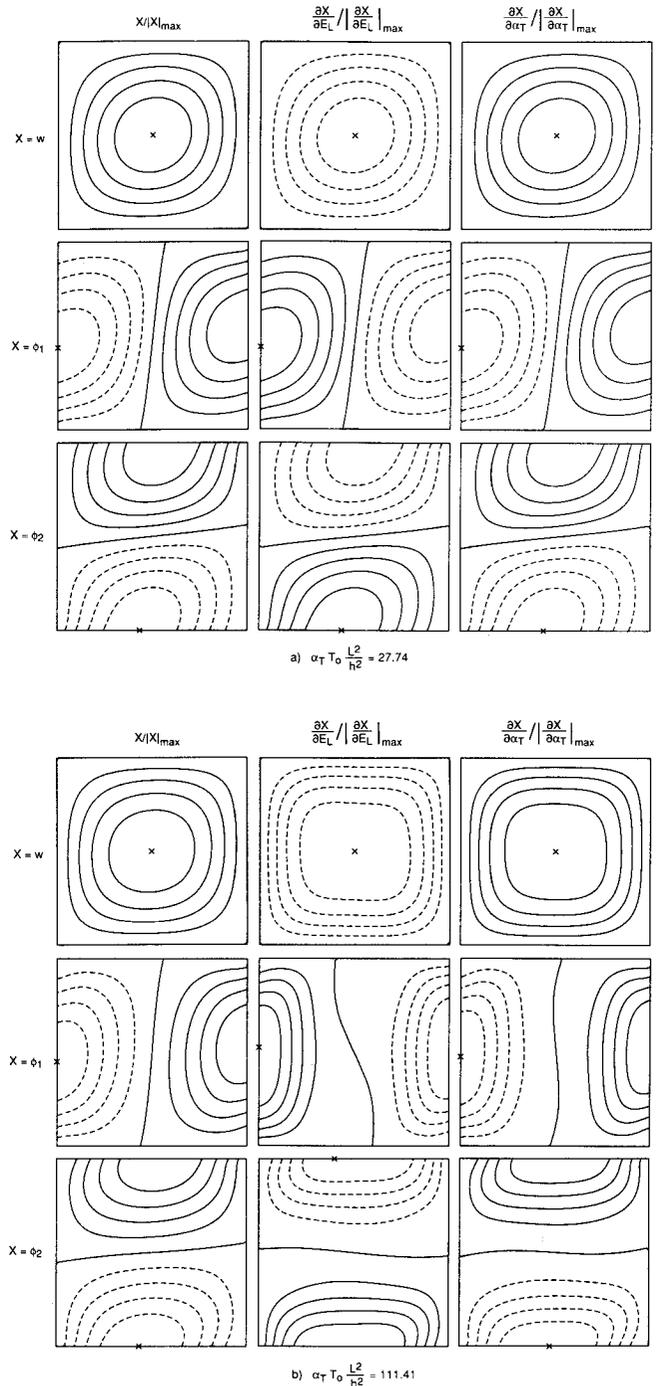
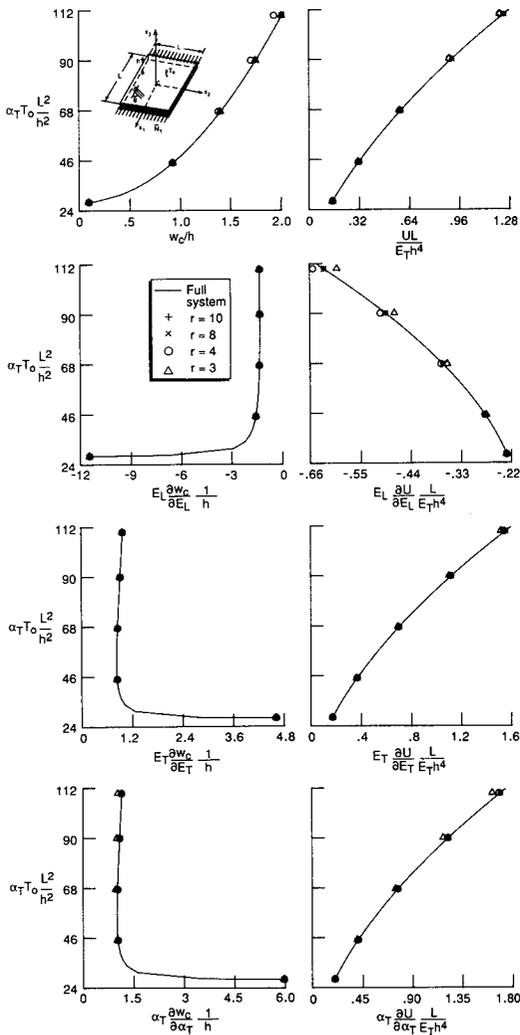


Fig. 6 Accuracy of the postbuckling response and sensitivity coefficients obtained by the reduced basis technique. Composite plate subjected to combined temperature increase and axial edge compression (see Fig. 1). $\bar{N}_1/\bar{N}_{1,cr} = 0.50$.

Fig. 7 Normalized contour plots for the generalized displacement components and their sensitivity coefficients. Composite plate subjected to combined temperature increase and axial edge compression (see Fig. 1). $\bar{N}_1/\bar{N}_{1,cr} = 0.50$. Spacing of contour lines is 0.2 and dashed lines denote negative contours. An \times denotes location of maximum absolute value of the component.

for example, Refs. 16, 17, and 18) can now be extended to the prediction of the sensitivity coefficients as well. When a new set of basis vectors, for approximating the response, is generated, the derivatives of each of these vectors with respect to the design variables (columns of the matrix $[\partial\Gamma/\partial d_i]$) are generated as well.

3) As pointed out in the preceding section on numerical studies, the computational time associated with the foregoing technique is considerably less than the direct application of Eqs. (2). This is particularly true when the reduced basis technique is used in generating the nonlinear response, and when the sensitivity coefficients are needed at several different values of the load parameters.

4) The foregoing techniques can be easily extended to the evaluation of the second-order sensitivity coefficients, as well as to structural reanalysis (analysis of modified structures). For the second order sensitivity coefficients the matrix of basis vectors is selected to be $[\bar{\Gamma}] = [[\Gamma][\partial\Gamma/\partial d_i][\partial^2\Gamma/\partial d_i^2]]$. In structural reanalysis, the set of basis vectors included in $[\bar{\Gamma}]$ are used for predicting the nonlinear response of the modified structure.

Concluding Remarks

An efficient reduced basis technique is presented for calculating the sensitivity of the nonlinear structural response to variations in the design variables. The analytical formulation

is based on a form of the moderate rotation, geometrically nonlinear theory of the structure. A total Lagrangian description is used and the structure is discretized by using two-field mixed finite element models. The equations governing the sensitivity coefficients are obtained by implicit differentiation of the discrete finite element equations of the structure. The vector of structural response and its sensitivity coefficients, with respect to design variables, are each expressed as a linear combination of a small number of basis (or global approximation) vectors and the Bubnov-Galerkin technique is used to approximate each of the finite element equations governing

the response and the sensitivity coefficients by a small number of algebraic equations in the amplitudes of these vectors. The path derivatives (derivatives of the response vector with respect to a path parameter) are used as basis vectors for approximating the response. A combination of the path derivatives and their derivatives with respect to the design variables is used for approximating the sensitivity coefficients.

The potential of the proposed technique is discussed and the effectiveness of the basis vectors used in approximating the sensitivity coefficients is demonstrated by means of two numerical examples of the nonlinear and postbifurcation responses of an eight-layer quasi-isotropic plate.

Acknowledgment

The present research is partially supported by NASA Cooperative Agreement NCCW-0011 and by an Air Force Office of Scientific Research Grant AFOSR-90-0369. The numerical studies were performed on the Cray Y-MP8/864 at the San Diego Supercomputer Center. The authors appreciate the encouragement of Samuel L. Veneri of NASA Headquarters and Spencer Wu of AFOSR.

Appendix A: Form of the Arrays in the Governing Discrete Equations of the Structure

The governing discrete equations of the structure, Eqs. (1), consist of both the constitutive relations and the equilibrium equations for the structure. The response vector $\{Z\}$ can be partitioned into the subvectors of stress parameters $\{H\}$ and the nodal displacements $\{X\}$ as follows:

$$\{Z\} = \begin{Bmatrix} H \\ X \end{Bmatrix} \quad (A1)$$

The different arrays in Eqs. (1) can be partitioned as follows:

$$[K] = \begin{bmatrix} -F & S \\ S^t & 0 \end{bmatrix} \quad (A2)$$

$$\{G(Z)\} = \begin{Bmatrix} M(X) \\ N(H, X) \end{Bmatrix} \quad (A3)$$

$$\{Q^{(1)}\} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \quad (A4)$$

$$\{Q^{(2)}\} = \begin{Bmatrix} \epsilon_T \\ 0 \end{Bmatrix} \quad (A5)$$

where $[F]$ is the flexibility matrix; $[S]$ is the linear strain-displacement matrix; $\{M(X)\}$ and $\{N(H, X)\}$ are the subvectors of nonlinear terms; $\{P\}$ is the vector of normalized applied mechanical loads; and $\{\epsilon_T\}$ is the vector of normalized thermal strains.

Appendix B: Evaluation of Path Derivatives and Their First Derivatives with Respect to d_i

The path derivatives in Eqs. (17) are obtained by successive differentiation of the governing finite element equations, Eqs. (1), with respect to the two parameters q_1 and q_2 , and solving the resulting system of linear algebraic equations. The recursion relations for the path derivatives can be written in the following compact form:

$$\left[K_{IJ} + \frac{\partial G_I}{\partial Z_J} \right] \frac{\partial^{s+t} Z_J}{\partial q_1^s \partial q_2^t} = R_I^{(s+t)} \quad (B1)$$

where I, J range from 1 to the total number of degrees of freedom in the model. The total number of the $(s + t)$ com-

Table B1

$s + t$	s	t	$R_I^{(s+t)}$
1	1	0	$Q_I^{(1)}$
2	2	0	$-\frac{\partial^2 G_I}{\partial Z_J \partial Z_L} \frac{\partial Z_J}{\partial q_1} \frac{\partial Z_L}{\partial q_1}$
	1	1	$-\frac{\partial^2 G_I}{\partial Z_J \partial Z_L} \frac{\partial Z_J}{\partial q_1} \frac{\partial Z_L}{\partial q_2}$
3	3	0	$-3 \frac{\partial^2 G_I}{\partial Z_J \partial Z_L} \frac{\partial Z_J}{\partial q_1} \frac{\partial^2 Z_L}{\partial q_1^2}$
	2	1	$-\frac{\partial^2 G_I}{\partial Z_J \partial Z_L} \left(\frac{\partial^2 Z_J}{\partial q_1^2} \frac{\partial Z_L}{\partial q_2} + 2 \frac{\partial^2 Z_L}{\partial q_1 \partial q_2} \frac{\partial Z_J}{\partial q_1} \right)$

binations is $(r - 1)$, where r is the number of basis vectors, and the explicit forms of the components of the right-hand sides, $R_I^{(s+t)}$, are given in Table B1.

In Table B1, a repeated uppercase Latin index denotes summation over its full range and G_I are bilinear (or quadratic) functions of Z_J . The expressions for the other right-hand sides not listed in Table B1 (e.g., for $s = 0, t = 1$, and $s = 0, t = 2$) are obtained from the expressions given in Table B1 by interchanging q_1 and q_2 .

The sensitivity derivatives of the path derivatives are obtained by differentiating Eqs. (B1) with respect to the design variables d_i . The resulting equations can be written in the following compact form:

$$\left[K_{IJ} + \frac{\partial G_I}{\partial Z_J} \right] \frac{\partial^{s+t+1} Z_J}{\partial q_1^s \partial q_2^t \partial d_i} = \frac{\partial R_I^{(s+t)}}{\partial d_i} - \left[\frac{\partial K_{IJ}}{\partial d_i} + \frac{\partial^2 G_I}{\partial Z_J \partial d_i} + \frac{\partial^2 G_I}{\partial Z_J \partial Z_K} \frac{\partial Z_K}{\partial d_i} \right] \frac{\partial^{s+t} Z_J}{\partial q_1^s \partial q_2^t} \quad (B2)$$

where d_i refers to any of the design variables and $s + t > 0$. For the case when $s = t = 0$, the sensitivity derivatives are given by Eqs. (2).

References

- Ryu, Y. S., Haririan, M., Wu, C. C., and Arora, J. S., "Structural Design Sensitivity Analysis of Nonlinear Response," *Computers & Structures*, Vol. 21, No. 1/2, 1985, pp. 245-255.
- Mroz, Z., Kamat, M. P., and Plaut, R. H., "Sensitivity Analysis and Optimal Design of Nonlinear Beams and Plates," *Journal of Structural Mechanics*, Vol. 13, Nos. 3/4, 1985, pp. 245-266.
- Haftka, R. T., and Mroz, Z., "First- and Second-Order Sensitivity Analysis of Linear and Nonlinear Structures," *AIAA Journal*, Vol. 24, No. 7, July 1986, pp. 1187-1192.
- Wu, C. C., and Arora, J. S., "Design Sensitivity Analysis and Optimization of Nonlinear Structural Response Using Incremental Procedure," *AIAA Journal*, Vol. 25, No. 8, Aug. 1987, pp. 1118-1125.
- Cardoso, J. B., and Arora, J. S., "Variational Method for Design Sensitivity Analysis in Nonlinear Structural Mechanics," *AIAA Journal*, Vol. 26, No. 5, May 1988, pp. 595-603.
- Tortorelli, D. A., Haber, R. B., and Lu, S. C.-Y., "Design Sensitivity Analysis for Nonlinear Thermal Systems," *Computer Methods in Applied Mechanics and Engineering*, Vol. 77, 1989, pp. 61-77.
- Arora, J. S., and Cardoso, J. E. B., "A Design Sensitivity Analysis Principle and Its Implementation into ADINA," *Computers and Structures*, Vol. 32, No. 3/4, 1989, pp. 691-705.
- Tortorelli, D. A., Haber, R. B., and Lu, S. C.-Y., "Adjoint Sensitivity Analysis for Nonlinear Dynamic Thermoelastic Systems," *AIAA Journal*, Vol. 29, No. 2, Feb. 1991, pp. 253-263.
- Phelan, D. G., Vidal, C., and Haber, R. B., "An Adjoint Variable Method for Sensitivity Analysis of Nonlinear Elastic Systems," *International Journal for Numerical Methods in Engineering*, Vol. 31, 1991, pp. 1649-1667.
- Noor, A. K., Tanner, J. A., and Peters, J. M., "Sensitivity of Tire Response to Variations in Material and Geometric Parameters,"

Finite Elements in Analysis and Design, Vol. 11, No. 1, May 1992.

¹¹Adelman, H. M., and Haftka, R. T., "Sensitivity Analysis of Discrete Structural Systems," *AIAA Journal*, Vol. 24, No. 5, May 1986, pp. 823-832.

¹²Haftka, R. T., and Adelman, H. M., "Recent Developments in Structural Sensitivity Analysis," *Structural Optimization I*, Springer-Verlag, Berlin/New York/Heidelberg, 1989, pp. 137-151.

¹³Haber, R. B., Tortorelli, D. A., Vidal, C. A., and Phelan, D. G., "Design Sensitivity Analysis of Nonlinear Structures—I: Large-Deformation Hyperelasticity and History-Dependent Material Response," *Structural Optimization: Status and Promise*, edited by M. P. Kamat, AIAA Series: *Progress in Astronautics and Aeronautics* (to appear).

¹⁴Arora, J. S., Lee, T. H., and Kumar, V., "Design Sensitivity Analysis of Nonlinear Structures—III: Shape Variation of Visco-plastic Structures," *Structural Optimization: Status and Promise*, AIAA Series: *Progress in Astronautics and Aeronautics* (to appear).

¹⁵Noor, A. K., "Recent Advances in Reduction Methods for Non-linear Problems," *Computers and Structures*, Vol. 13, No. 1/2, 1981, pp. 31-44.

¹⁶Noor, A. K., "On Making Large Nonlinear Problems Small," *Computer Methods in Applied Mechanics and Engineering*, Vol. 34, Nos. 1-3, 1982, pp. 955-985.

¹⁷Noor, A. K., and Peters, J. M., "Recent Advances in Reduction Methods for Instability Analysis of Structures," *Computers and Structures*, Vol. 16, No. 1-4, Jan. 1983, pp. 67-80.

¹⁸Noor, A. K., and Peters, J. M., "Multiple-Parameter Reduced Basis Technique for Bifurcation and Postbuckling Analyses of Composite Plates," *International Journal for Numerical Methods in Engineering*, Vol. 19, No. 12, 1983, pp. 1783-1803.

¹⁹Noor, A. K., and Camin, R. A., "Symmetry Considerations for Anisotropic Shells," *Computer Methods in Applied Mechanics and Engineering*, Vol. 9, No. 3, 1976, pp. 317-335.

²⁰Noor, A. K., and Andersen, C. M., "Mixed Models and Reduced/Selective Integration Displacement Models for Nonlinear Shell Analysis," *International Journal for Numerical Methods in Engineering*, Vol. 18, No. 10, 1982, pp. 1429-1454.

²¹Choi, K. K., and Twu, S. L., "Equivalence of Continuum and Discrete Methods of Shape Design Sensitivity Analysis," *AIAA Journal*, Vol. 27, Oct. 1989, pp. 1418-1424.

Recommended Reading from
Progress in Astronautics and Aeronautics

MECHANICS AND CONTROL OF LARGE FLEXIBLE STRUCTURES

J.L. Junkins, editor

This timely tutorial is the culmination of extensive parallel research and a year of collaborative effort by three dozen excellent researchers. It serves as an important departure point for near-term applications as well as further research. The text contains 25 chapters in three parts: Structural Model-

ing, Identification, and Dynamic Analysis; Control, Stability Analysis, and Optimization; and Controls/Structure Interactions: Analysis and Experiments. 1990, 705 pp, illus, Hardback, ISBN 0-930403-73-8, AIAA Members \$69.95, Nonmembers \$99.95, Order #: V-129 (830)

Place your order today! Call 1-800/682-AIAA



American Institute of Aeronautics and Astronautics
Publications Customer Service, 9 Jay Gould Ct., P.O. Box 753, Waldorf, MD 20604
Phone 301/645-5643, Dept. 415, FAX 301/843-0159

Sales Tax: CA residents, 8.25%; DC, 6%. For shipping and handling add \$4.75 for 1-4 books (call for rates for higher quantities). Orders under \$50.00 must be prepaid. Please allow 4 weeks for delivery. Prices are subject to change without notice. Returns will be accepted within 15 days.